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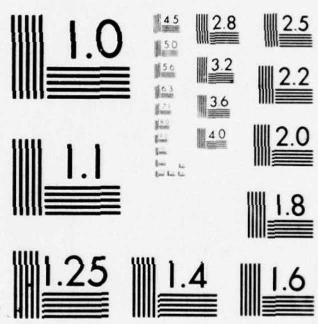

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SENSITIVITY ANALYSIS IN NONLINEAR PROGRAMMING USING FACTORABLE SYMBOLIC INPUT

by

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Anura de Silva
Garth P. McCormick

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1. Introduction

This paper develops a methodology to utilize the input to nonlinear programs in symbolic factorable form to perform first order sensitivity analysis on the solution vector $x(y) \epsilon E^n$ of a general nonlinear program, where $y \epsilon E^k$ is the parametric vector. In Section 2 we discuss the key results of sensitivity analysis theory in nonlinear programming. In Section 3 we develop the formulae underlying the new methodology to utilize the symbolic factorable input to perform sensitivity analysis on the solution vector $x(y)$.

The above are implemented via the SUMT algorithm [4] for nonlinear programming which uses a penalty function technique. Section 4 briefly discusses the penalty function-based implementation of the sensitivity analysis methodology. An extrapolation result based on the penalty function parameter r is stated and incorporated in the computer code.

2. Relevant Results from NLP Sensitivity Theory

Consider the problem

$$\left.\begin{array}{l} \underset{x \in E^n}{\text{MIN}} \quad F(x,y) \\ \quad \text{s.t.} \quad G(x,y) \geqq 0 \\ \quad\quad\quad\; H(x,y) = 0 \\ \text{where} \quad G : E^n \times E^k \to E^m \\ \quad\quad\quad\; H : E^n \times E^k \to E^p \end{array}\right\} \text{Problem } M(y)$$

By sensitivity information we mean the rates of change of the (locally) optimal objective function value and the corresponding optimizing vector with respect to changes in the parametric vector. If $x(y)$ is an n-vector and y a k-vector, the first order sensitivity information for Problem M(y) is:

(a) The rate of change of the solution value with respect to the parametric vector, $D_y F[x(y),y] \in E^k$.

(b) The rate of change of the optimizing vector with respect to the parametric vector, $\frac{dx(y)}{dy}$, where

$$\frac{dx(y)}{dy} = \begin{bmatrix} \frac{\partial x_1(y)}{\partial y_1} & \cdot\;\cdot & \frac{\partial x_1(y)}{\partial y_k} \\ \vdots & & \vdots \\ \frac{\partial x_n(y)}{\partial y_1} & \cdot\;\cdot & \frac{\partial x_n(y)}{\partial y_k} \end{bmatrix} \tag{1}$$

The first order sensitivity results for the general nonlinear programming problem were first stated by Fiacco and McCormick in [3], Theorem 6, where they give conditions under which a smooth (once continuously differentiable, abbreviated OCD) trajectory of the optimizing Kuhn-Tucker triple $[x(y),u(y),w(y)]$ exist for y close to an initial value y_0 . The u and w are the Lagrange multiplier vectors associated with the inequality

constraints G and equality constraints H of Problem M(y), in the Lagrangian,

$$L(x,y,u,w) = F(x,y) - \sum_{i=1}^{m} u_i G_i(x,y) + \sum_{j=1}^{p} w_j H_j(x,y) \qquad (2)$$

where $u_i, G_i; i = 1,\ldots,m$ and $w_j, H_j: j = 1,\ldots,p$ are the components of the mappings G and H and the Lagrange multipliers u and w. Here, u(y) and w(y) are the values of these multipliers associated with the optimal value x(y). They assume the following to hold; (a) twice continuous differentiability (abbreviated TCD) of L(x,y,u,w) in (x,y) in a neighborhood of $(x(y_0),y_0)$, (b) second order sufficiency conditions (abbrev. SOSC) at $(x(y_0),y_0)$, (c) linear independence of binding constraints (abbrev. LIB) at $(x(y_0),y_0)$, (d) strict complementary slackness (abbrev. SCS). Here, (c) and (d) are regularity conditions they invoke to obtain the key result in a useful form. In a more recent paper Fiacco [2] obtains a similar result for somewhat more general parametric problem, such as M(y).

The crux of the theory is the application of the implicit function theorem (see [6] for a general treatment of the implicit function theorem) to the first order Kuhn-Tucker system of M(y), which is

$$\begin{aligned} \nabla_x L(x,y,u,w) &= 0 \\ u_i G_i(x,y) &= 0, \; i = 1,\ldots,m \\ H_j(x,y) &= 0, \; j = 1,\ldots,p. \end{aligned} \qquad (3)$$

Consider a system of equations

$$D(x,y) = 0, \; x\varepsilon E^n \text{ and } y\varepsilon E^k \qquad \text{Problem } P(y),$$

where $D : E^n \times E^k \to E^\ell$, ℓ being the number of scalar valued functions in D, which are assumed OCD. This ensures the existence of the Jacobians of D(x,y) with respect to x and y, denoted J_xD and J_yD, respectively. With the sensitivity analysis context in view, the implicit function theorem could be stated as follows:

Theorem 1. Suppose $D : E^n \times E^k \to E^\ell$; $(x(y_0),y_0)$ is a zero of $D(x,y)$; $D(x,y)$ is OCD near $(x(y_0),y_0)$; $J_x D$ is nonsingular at $(x(y_0),y_0)$.

Then, there exist open sets $S_x(x(y_0)) \subset E^n$ and $S_y(y_0) \subset E^k$ such that for any $y \in \bar{S}_y(y_0)$ there exists a unique solution for x in terms of y , i.e., $x = x(y) \in S_x(x(y_0))$, which is continuous in y . Moreover $x(y)$ is OCD at y_0 and

$$\nabla_y x(y_0) = - \left[J_x D_0\right]^{-1} \left[J_y D_0\right] ,$$

where the subscript 0 on D denotes evaluation at $(x(y_0),y_0)$.

If $P(y)$ is compared with (3) the role of Theorem 1 in the context of sensitivity analysis of the solution vector becomes evident. The assumed TCD of the problem functions gives the necessary OCD for the system in (3). The additional conditions of linear independence of the binding constraint gradients and SCS were assumed by Fiacco and McCormick [3] and Fiacco [2] to produce desirable results such as the uniqueness of the Lagrange multipliers $[u(y),w(y)]$ associated with the optimal $x(y)$ and the invariance of the set of inequality constraints binding at $x(y_0)$ in a neighborhood of y_0 . The following representation for the Kuhn-Tucker triple,

$$X \triangleq (x,u,w)$$

$$\text{and} \quad X(y) \triangleq [x(y),u(y),w(y)]$$

will be adopted for notational convenience. For the same reason, the first order Kuhn-Tucker system of (3) will be denoted as follows.

$$R(x,u,w,y) = 0 \quad \text{or} \quad \bar{R}(X,y) = 0 . \tag{3b}$$

Differentiating (3b) partially with respect to X yields the Jacobian of (3b) or (3) with respect to X ,

$$J_X\bar{R}(X,y) = \left[\begin{array}{c|ccc|ccc}
\nabla^2_{xx}L & -\nabla_x G_1, & \cdots & -\nabla_x G_m & \nabla_x H_1, & \cdots & \nabla_x H_p \\
\hline
u_1\nabla^T_x G_1 & G_1 & & 0 & & & \\
\vdots & & \ddots & & & 0 & \\
u_m\nabla^T_x G_m & 0 & & G_m & & & \\
\hline
\nabla^T_x H_1 & & & & & & \\
\vdots & & 0 & & & 0 & \\
\nabla^T_x H_p & & & & & &
\end{array}\right] \tag{4}$$

Differentiating (3b) partially with respect to y yields the Jacobian with respect to this parametric vector,

$$J_y\bar{R}(X,y) = \left[\nabla^2_{yx}L^T,\ u_1\nabla_y G_1, \ldots, u_m\nabla_y G_m,\ \nabla_y H_1, \ldots, \nabla_y H_p\right]^T . \tag{5}$$

At a Kuhn-Tucker point, (3b) could be written as

$$\bar{R}[X(y),y] = 0 . \tag{3c}$$

Totally differentiating (3c) with respect to y , the following relationship is obtained.

$$J_X\bar{R}(X(y),y)\ \frac{dX(y)}{dy} + J_y\bar{R}(X(y),y) = 0 . \tag{6}$$

The principal result immediately follows.

$$\frac{dX(y)}{dy} = -\ J_X\bar{R}(X(y),y)^{-1} \cdot J_y\bar{R}(X(y),y) . \tag{7}$$

Equation (7) is almost the same as the result stated under Theorem 1 but is stated with respect to the entire Kuhn-Tucker triple, X . Note that (7) is stated for y in a neighborhood of y_0 , whereas the result of Theorem 1 was only at y_0 . This is possible due to the slightly stronger assumptions made by Fiacco and McCormick [3] and Fiacco [2].

The above result is the basis for the development of a methodology for sensitivity analysis in nonlinear programming. The computational technique used here is the penalty function method. The same result and technique were used by Armacost and Mylander [1] to compute (1), the principal difference in their methodology being that they approximate $J_y\bar{R}$ by central differencing techniques, and compute all other derivatives manually. The relevant formulae for the penalty function implementation using the logarithmic barrier-quadratic loss penalty function of the SUMT algorithm [4] are discussed in Section 3.

3. Sensitivity Analysis With Symbolic Input

The factorable input technique and computer code to compute gradient vectors and Hessian matrices of problem functions was developed by McCormick [5] to facilitate the use of nonlinear programming computer codes. The code itself has since been extended by de Silva and McCormick to accept input in a symbolic format. This symbolic code has been used to solve several general nonlinear programming problems and the ease with which a problem input can be supplied or modifications done is very encouraging. A sensitivity analysis option turned out to be a natural extension to the symbolic computer code.

A reduced version of the matrix in (1) which provides substantial computational economy, results from the formulae which are developed below; a brief discussion of the factorable input philosophy precedes this development.

In the factorable input method, the problem functions are built up recursively as sums, or sums of pairwise products, of simple transformations of previously defined functions. Initially, these simple transformations

are applied to the original problem variables x_i, $i = 1,\ldots,n$, and the recursive application of the above procedure (shown mathematically in Equation (8)) generates new functions X^i, $i = n+1,\ldots,N$.

McCormick [5] gives a detailed exposition illustrated by an example. He sets $X^i = x_i$, $i = 1,\ldots,n$ and refers to the entire set of X^i, $i = 1,\ldots,N$ as concomitant variable functions (abbrev. CVFs). The general equation for the creation of a new CVF is:

$$X^i(x) = \sum_{p=1}^{i-1} T_p^i[X^p(x)] + \sum_{p=1}^{i-1} \sum_{q=1}^{p} V_{q,p}^i[X^p(x)] \cdot U_{p,q}^i[X^q(x)] \; . \qquad (8)$$

In the presence of a parametric vector y, the equation becomes

$$X^i(x,y) = \sum_{p=1}^{i-1} T_p^i[X^p(x,y),y]$$
$$+ \sum_{p=1}^{i=1} \sum_{q=1}^{r} V_{q,p}^i[X^p(x,y),y] \cdot U_{p,q}^i[X^q(x,y),y] \; . \qquad (9)$$

The terms in the first summation of (8) or (9) are referred to as <u>separable</u> terms comprising the new CVF $X^i(\cdot)$ and the terms of the second summation of pairwise products are called <u>quadratic</u> terms. A point of note in (9) is that the arguments of the simple transformation T_p^i, $V_{q,p}^i$ or $U_{p,q}^i$ contain y directly as well as indirectly via the argument CVF. The direct presence of the parametric vector y is restricted to, at most, two components because all the simple transformations used in the methodology and code contain, at most, two constant terms. The parametric components are a subset of these constants.

An examination of (7) suggests that second partial derivatives of the problems functions with respect to x and the cross second partial derivatives with respect to x and y are required. This is because $\bar{R}$ already contains first partial derivatives with respect to x. What is developed below will yield the matrix in (1) reduced by postmultiplying it

with a known column vector $\beta \varepsilon E^k$ or premultiplying it with a known row vector $\sigma \varepsilon E^n$. The postmultiplication produces an n-vector, each component being a weighted sum of the k partial derivatives of a particular $x_i(y)$, with respect to the components of the parametric vector. The weights are the components of $\beta \varepsilon E^k$. The ith component of this resulting n-vector can be looked upon as the "total sensitivity" of the variable $x_i(y)$ $(i=1,\ldots,n)$ with respect to all the y_j , $j = 1,\ldots k$, weighted according to β_j , $j = 1,\ldots,k$. The rth component of the n-vector is $\sum_{j=1}^{k} \frac{x_r(y)}{\partial y_j} \beta_j$. The premultiplication produces a k-component vector, the jth component being the "total sensitivity" of a composite of all n variables $x_i(y)$, $i = 1,\ldots,n$, weighted according to σ_i , $i = 1,\ldots,n$, with respect to $y_j (j=1,\ldots,k)$. The ℓth component of this k-vector would be $\sum_{i=1}^{n} \frac{\partial x_i}{\partial y_\ell} \sigma_i$.

For clarity of exposition, (9) will be simplified below to a separable portion of one term or a quadratic term of one product.

Consider the separable case first. Suppose

$$X^i(x,y) = T[X^P(x,y),y] \quad . \tag{10}$$

Letting

$$\dot{T} \triangleq \dot{T}\,[X^P(x,y),y] \triangleq \frac{\partial T[X^P(x,y),y]}{\partial X^P} , \tag{11}$$

we have

$$\nabla_x X^i(x,y) = \nabla_x X^P(x,y) \cdot \frac{\partial T[X^P(x,y),y]}{\partial X^P} . \tag{12a}$$

$$= \dot{T} \cdot \nabla_x X^P(x,y) . \tag{12b}$$

In (12b) $\dot{T}$ is readily available since all transformations are simple ones, and $\nabla_x X^p(x,y)$ is built up inductively from its argument CVF.

The expressions for $\nabla^2_{xx} X^i(x,y)$ are already developed and coded in [5], and will not be repeated here. Note that

$$\nabla^2_{yx} X^i(x,y) = \nabla_y \nabla_x X^i(x,y) \tag{13a}$$

$$= D_y D_x \; T[X^p(x,y),y] \; . \tag{13b}$$

In (13b) D denotes total differentiation, although in the previous step the partial differentiation ∇ was used. This is due to the fact that the partial differentiation in (13a) would capture the total variation due to x and y of the expression in (12a), but not so in (13b). Continuing the differentiation with respect to y of (12b), as in (13b), we get

$$\nabla^2_{yx} X^i(x,y) = \nabla^2_{yx} X^p(x,y) \cdot \dot{T} + \nabla_x X^p(x,y) \left\{ \ddot{T} \; [X^p(x,y),y] \; \nabla^T_y X^p(x,y) + \nabla^T_y \; \dot{T} \right\} . \tag{14}$$

In the above, $\ddot{T} \triangleq \ddot{T} \; [X^p(x,y),y] \triangleq \dfrac{\partial^2 T[X^p(x,y),y]}{\partial (X^p)^2}$. (15)

In (14), $\dot{T}$ and $\ddot{T}$ are readily available scalar quantities. The quantity $\nabla^T_y \; \dot{T} \; [X^p(x,y),y]$ is the true partial derivative unlike in (13b). This is easy to compute because y enters the $\dot{T}$ argument directly, at most, in two nonzero components. The expression for $\nabla_x X^p(x,y)$ has been derived already in (12a). The only remaining terms are $\nabla^2_{yx} X^p(x,y)$ which is built up inductively, and $\nabla^T_y X^p(x,y)$ which is discussed below.

Now

$$\nabla_y X^i(x,y) = D_y \; T[X^p(x,y),y]$$
$$= \nabla_y \; T[X^p(x,y),y] + \dot{T} \cdot \nabla_y X^p(x,y) \; . \qquad (16)$$

Note $\nabla_y T(X^p(x,y),y)$ is the true partial derivative and will have at most two nonzero components in E^k which are easily computable. Thus (16) permits the computation of the $\nabla_y^T X^p(x,y)$ term of (14) by induction. Since the formulae of (12b), (14), (16) are initially applied to the original problem variables $X_i = x^i$, $i = 1,\ldots,n$, we have, for $p \leq n$:

$$\nabla_x X^i(x,y) = e_p \cdot \dot{T} \qquad (17)$$

where e_j is a unit vector εE^n with unity in the jth position.

Also,

$$\nabla^2_{yx} X^i(x,y) = \nabla^2_{yx} X^p + e_p[\ddot{T} \; \nabla^T X^p + \nabla_y^T \; \dot{T}]$$
$$= e_p \; \nabla_y^T \; \dot{T} \; . \qquad (18)$$

Finally,

$$\nabla_y X^i(x,y) = \nabla_y T + \nabla_y X^p \; \dot{T}$$
$$= \nabla_y T \; . \qquad (19)$$

On page 6 , we spoke of dimensionally reducing the matrix of (1) in two separate ways. The case of premultiplying by $\alpha \; \varepsilon \; E^n$ is considered below.

Thus, from (17) and (18) letting α_p denote the pth component of α , we get

$$\alpha^T \nabla_x X^i(x,y) = \alpha_p \cdot \dot{T} \; . \qquad (20)$$

$$\alpha^T \nabla^2_{yx} X^i(x,y) = \alpha_p \nabla_y^T \; \dot{T} \; . \qquad (21)$$

The summation of (21) will have, at most, two nonzero components due to the nature of the simple transformations used. Next, consider the case of $p > n$. Here,

$$\alpha^T \nabla_x X^i(x,y) = \alpha^T \nabla_x X^p(x,y) \cdot \dot{T} \cdot \tag{22}$$

Next,

$$\alpha^T \nabla^2_{yx} X^i(x,y) = \alpha^T \nabla^2_{yx} X^p(x,y) \cdot \dot{T}$$

$$+ \alpha^T \nabla_x X^p(x,y) \left\{ \ddot{T} \nabla_y^T X^p(x,y) + \nabla_y^T \dot{T} \right\} . \tag{23}$$

Equation (22) works purely inductively and does not require the deliberate introduction of α , as this has been done in (20). It is a scalar quantity for each i . The first term on the right-hand side of (23) is also an inductive term. As far as the second term is concerned, (the scalar multiplier $\alpha^T \nabla_x X^p(x,y)$, the only presence of α there,) it is built up inductively. The expression within the curly bracket is a k-vector, the first term being an inductive one as demonstrated in (16) and the second possessing, at most, two nonzero components.

In summary, we have a scalar quantity for the reduced version of the gradient vector of each CVF i , and (20) (or (17), if $p \leq n$) has to be repeated for every p corresponding to the separable terms of X^i , and added together. The reduced second cross partial of each CVF i (as shown in (21) or (18) depending on whether $p \gtrless n$) is a k-vector for each component p , which must be added together as before.

The quadratic components of (9) will be addressed next. Considering the creation of an X^i , $i = n+1,\ldots,N$, using a single product, we have

$$X^i(x,y) = V[X^p(x,y),y] * U[X^q(x,y),y] \ , \tag{24}$$

then,

$$\nabla_x X^i(x,y) = D_x V[X^p(x,y),y] * U[X^q(x,y),y]$$

$$+ V[X^p(x,y),y] * D_x U[X^q(x,y),y] \tag{25}$$

$$= \dot{V}U \cdot \nabla_x X^p(x,y) + V\dot{U} \cdot \nabla_x X^q(x,y) \ . \tag{26}$$

Now $\dot{V}$, $\dot{U}$ are defined analogously to $\dot{T}$ (see (11)). Since V , $\dot{V}$, U , $\dot{U}$ are scalar quantities, the premultiplication by $\alpha \in E^n$ could be done as in (20) and (22) depending on whether p and q are $\leq$ or $>$ n . The separate terms in (25) must be added together. The result is

$$\alpha^T \nabla_x X^i(x,y) = \dot{V}U \; \alpha^T \nabla_x X^p(x,y) + V\dot{U} \; \alpha^T \nabla_x X^q(x,y) \; . \tag{26}$$

When p (or q) is $> n$, $\alpha^T \nabla_x X^p$ (or $\alpha^T \nabla_x X^q$) enters purely inductively. Otherwise, the term simplifies to $\dot{V}U \; \alpha_p$ (or $V\dot{U} \; \alpha_q$). The second cross partial derivative is obtained by differentiating (26) with respect to y . The arguments of X^p and X^q are omitted for convenience of representation. Finally,

$$\begin{aligned} \nabla^2_{yx} X^i(x,y) = U &\left\{ \dot{V} \cdot \nabla^2_{yx} X^p + \nabla_x X^p [\ddot{V} \cdot \nabla^T_y X^p + \nabla^T_y \dot{V}] \right\} \\ &+ \dot{V} \cdot \nabla_x X^p \cdot D_y U \\ &+ V \left\{ \dot{U} \cdot \nabla^2_{yx} X^q + \nabla_x X^q [\ddot{U} \cdot \nabla^T_y X^q + \nabla^T_y \dot{U}] \right\} \\ &+ \dot{U} \cdot \nabla_x X^q \cdot D_y V \; . \end{aligned} \tag{27}$$

Premultiplying (27) by $\alpha \in E^n$, and rearranging,

$$\begin{aligned} \alpha^T \nabla^2_{yx} X^i(x,y) = U &\left\{ \alpha^T \nabla^2_{yx} X^p \cdot \dot{V} + \alpha^T \nabla_x X^p [\ddot{V} \cdot \nabla^T_y X^p + \nabla^T_y \dot{V}] \right\} \\ &+ V \left\{ \alpha^T \nabla^2_{yx} X^q \cdot \dot{U} + \alpha^T \nabla_x X^q [\ddot{U} \cdot \nabla^T_y \dot{U}] \right\} \\ &+ \alpha^T \nabla_x X^p \cdot D_y U \cdot \dot{V} + \alpha^T \nabla_x X^q \cdot \nabla_y D \cdot \dot{U} \; . \end{aligned} \tag{28}$$

The evaluation of the portions within the curly brackets in (28) is identical to the evaluation of the cross second partials in the separable analysis, i.e., the right-hand side of (23). The last two terms of (26) contain two quantities $\alpha^T \nabla_x X^p$ and $\alpha^T \nabla_x X^q$ which are developed by induction as demonstrated by (12) and (20). The quantities $D_y U$ and $D_y V$

are analogous to $D_y T$ whose computation is shown in (16). The only remaining expression to be developed is that for $\nabla_y X^i$,

$$\begin{aligned}\nabla_y X^i(x,y) &= D_y\{V[X^p(x,y),y]\ U[X^q(x,y),y]\} \\ &= U\{\dot{V}\cdot\nabla_y X^p+\nabla_y V\} + V\{\dot{U}\ \nabla_y X^q+\nabla_y U\}\ .\end{aligned} \tag{29}$$

Examination of (23) and (28) shows that it is not necessary to premultiply the expressions of (16) and (29) by α^T , since $\nabla_y X$ never gets directly premultiplied by it.

The above Equations (20) through (29) have developed the necessary components to perform the computations necessary to produce a row k-vector $\alpha^T[dx(y)/dy]$. The next section describes how these are implemented via a penalty function method.

4. Computation of Sensitivity Results

The SUMT computer code [4] is used in conjunction with the factorable symbolic sensitivity analysis code to yield $[\alpha^T dx(y)/dy]$. The code uses an interior-exterior penalty function with a single parameter r , and generates a sequence of subproblems which will be denoted as $M(y,r)$, which tend to $M(y)$ as $r \to 0$. In straight nonlinear optimization SUMT uses an extrapolation based on r to estimate the relevant values at $r = 0$. A similar extrapolation technique is used here to provide first order estimates of $[\alpha^T dx(y)/dy]$.

A logarithmic barrier quadratic loss penalty function is considered below. For $M(y)$, this penalty function is

$$P[x,y,r] = F(x,y) - r \sum_{i=1}^{m} \ln G_i(x,y) + \sum_{j=1}^{p} \frac{H_j^2(x,y)}{r}\ . \tag{30}$$

The problem solved at the ℓth stage of the sequential procedure, for a value $r = r^\ell$ is

$$\min_{x\varepsilon R^0} P[x,y,r^\ell] \qquad \text{Problem } M(r,y)$$

where

$$R^0 \equiv \{x \mid G_i(x,y) > 0\ ,\ i=1,\ldots,m\}\ .$$

The optimality condition is that the gradient with respect to x vanishes, i.e.,

$$\nabla_x P[x,y,r^\ell] = 0 . \tag{31}$$

At local optima, for a general value of r we can write

$$\nabla_x P[x(y,r),y,r] \equiv 0 . \tag{32}$$

Applying this to (30),

$$\nabla_x F[x(y,r),y] - \sum_{i=1}^{m} \frac{r}{G_i[x(y,r),y]} \cdot \nabla_x G_i[x(y,r),y]$$
$$+ \sum_{i=1}^{p} \frac{2\, H_j[x(y,r),y]}{r} \nabla_x H_j[x(y,r),y] \equiv 0 . \tag{33}$$

Differentiating (33) partially with respect to y, we get

$$\nabla^2_{yx} P[x(y,r),y,r]$$
$$= \nabla^2_{yx} F[x(y,r),y] - \sum_{i=1}^{m} \nabla^2_{yx} G_i[x(y,r),y] \cdot \frac{r}{G_i[x(y,r),y]}$$
$$+ \sum_{i=1}^{m} \frac{r}{G_i[x(y,r),y]^2} \cdot \nabla_x G_i[x(y,r),y] \cdot \nabla^T_y G_i[x(y,r),y]$$
$$+ \sum_{j=1}^{p} \nabla^2_{yx} H_j[x(y,r),y] \cdot \frac{2\, H_j[x(y,r),y]}{r}$$
$$+ \sum_{j=1}^{p} \left(\frac{2}{r}\right) \nabla_x H_j[x(y,r),y] \cdot \nabla^T_y H_j[x(y,r),y] . \tag{34}$$

Differentiating (33) totally with respect to y,

$$\nabla^2_{xx} P[x(y,r),y,r] \frac{dx(y,r)}{dy} + \nabla^2_{yx} P[x(y,r),y,r] \equiv 0 . \tag{35}$$

The argument variables within the square brackets will be omitted when referring to the terms in the above.

$\nabla^2_{xx}P$ has been developed elsewhere by McCormick [5]. Those for $\nabla^2_{xy}P$ are developed using the Equation (34). Thus,

$$\frac{d(x(y,r))}{dy} = - \nabla^2_{xx}P[x(y,r),y,r]^{-1} \cdot \nabla^2_{yx}P[x(y,r),y,r] \ . \tag{36}$$

The similarity between (7) and (36) is to be noted. Suppose (36) is premultiplied by some $\sigma \varepsilon E^n$. Then, we get

$$\sigma^T \frac{dx(y,r)}{dy} = - \alpha^T \cdot \nabla^2_{yx}P[x(y,r),y,r] \ , \tag{37}$$

where

$$\alpha^T = \sigma^T \cdot \nabla^2_{xx}P[x(y,r),y,r]^{-1} \ . \tag{38}$$

If σ is a known vector, α is determined because the Hessian of the penalty function with respect to x is known by [5]. Then (37) is determined by premultiplying (34) by this α , and using (20), (21), (22), (23), (27) and (29) to determine the individual components of the resulting equation. This can be done because every problem function F , G_i ; $i = 1,\ldots,m$, H_j ; $j = 1,\ldots p$, can be identified with a CVF X^i , $i = 1,\ldots,n$, $n+1,\ldots,N$.

The final aspect remaining to be discussed in the context of the sensitivity methodology is the improvement of the estimate of $\sigma^T \frac{dx(y,r)}{dy}$ by extrapolating on r , at the ℓth sequential unconstrained problem, to $r = 0$. A Taylor series approximation applied at the ℓth and $(\ell-1)$th subproblems yields

$$\frac{dx(y,r^{\ell})}{dy} \doteq \frac{dx(y,0)}{dy} + r^{\ell} \frac{d}{dr}\left[\frac{dx(y,0)}{dy}\right] . \tag{39}$$

$$\frac{dx(y,r^{\ell-1})}{dy} \doteq \frac{dx(y,0)}{dy} + r^{\ell-1} \frac{d}{dr}\left[\frac{dx(y,0)}{dy}\right] . \tag{40}$$

Eliminating the last term in (39) and (40)

$$\frac{d(x(y,0))}{dy}\left(r^{\ell-1} - r^{\ell}\right) \doteq r^{\ell-1} \cdot \frac{dx(y,r^{\ell})}{dy} - r^{\ell} \cdot \frac{dx(y,r^{\ell-1})}{dy} \tag{41}$$

from which, (letting $r^{\ell-1} = c \cdot r^{\ell}$) we get

$$\frac{dx(y,0)}{dy} \doteq \frac{1}{c-1} \cdot \left[c \frac{dx(y_1 r^{\ell})}{dy} - \frac{dx(y_1 r^{\ell-1})}{dy}\right]. \tag{42}$$

Hence by performing the computation for successive subproblems when r becomes sufficiently small, (42) can be used to estimate the value for the original problem, i.e., $r = 0$.

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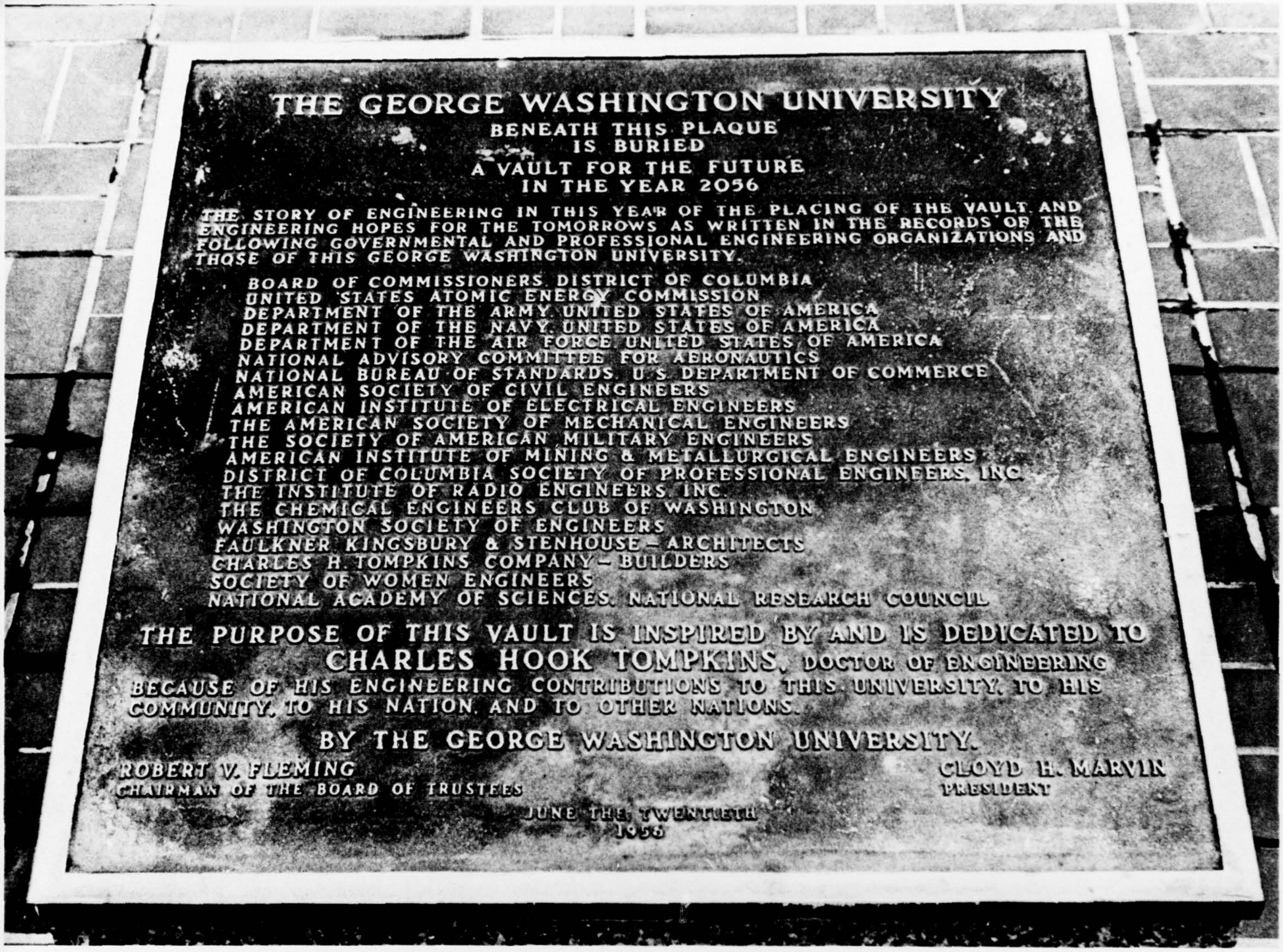

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